On the Apparent Disc and on the Shadow of an Ellipsoid.
By A. Marth.

If k denotes the equatorial radius and $e = \sin \epsilon_0$ the eccentricity of a planet's ellipsoidal surface, so that the polar radius is $k \cos \epsilon_0$, the outline of the planet's disc, as seen from a point in planetocentric latitude B above the plane of the planet's equator and at the distance Δ, will be an ellipse, the semi-axes a and b of which are

$$a = \frac{k}{\Delta}\omega'' \qquad \omega'' = \frac{1}{\text{arc } 1''}$$
$$b = a\sqrt{1 - e^2 \cos^2 B}$$
$$= a \cos \epsilon,$$

If, further,

$$\left.\begin{aligned} \sin \epsilon &= \sin \epsilon_0 \cos B. \\ \cos \epsilon \sin \nu &= \sin \epsilon_0 \sin B \\ \cos \epsilon \cos \nu &= \cos \epsilon_0, \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (1)$$

the proportion $\cos \epsilon$ of the axes of the apparent disc is found from the proportion $\cos \epsilon_0$ of the axes of the ellipsoid by

$$\cos \epsilon = \cos \epsilon_0 \sec \nu, \text{ where } \tan \nu = \tan \epsilon_0 \sin B,$$

and $\cos \epsilon_0$ from $\cos \epsilon$ by

$$\cos \epsilon_0 = \cos \epsilon \cos \nu, \text{ where } \sin \nu = \tan \epsilon \tan B.$$

These simple formulæ for the outline of the apparent disc are only valid when the distance Δ in comparison with k is so great that terms of the second order $\frac{k^2}{\Delta} \ldots$ may be neglected. If that is not the case, strict formulæ must be substituted.

Let λ and β denote the planetocentric longitude and latitude, referred to the plane of the planet's equator, of a point on the ellipsoidal surface, $k\rho$ its linear distance from the centre of the ellipsoid, and L B Δ the corresponding coordinates of the point of view, so that L B and $k \cos \nu$ are the coordinates of the centre point of the apparent disc. Further, let $p - P = \pi$ denote the position angle of the surface point at the apparent centre of the disc, reckoned from the direction P of the polar axis, σ the planetocentric angular distance or the angle between $k\rho$ and Δ, Δ' the linear distance between the surface point and the point of view, and s the angle between Δ' and Δ. The values of π σ s Δ' are found by

$$\left.\begin{aligned} \sin \sigma \sin \pi &= \cos \beta \sin (L - \lambda) \\ \sin \sigma \cos \pi &= -\cos \beta \cos (L - \lambda) \sin B + \sin \beta \cos B \\ \cos \sigma &= \cos \beta \cos (L - \lambda) \cos B + \sin \beta \sin B \\ \Delta' \sin s &= k\rho \sin \sigma \\ \Delta' \cos s &= \Delta - k\rho \cos \sigma. \end{aligned}\right\} \quad \ldots \quad (2)$$

Instead of the latitudes β and B, it will be advantageous to introduce latitudes β' and B$'$ connected with them by

$$\tan\beta' = \tan\beta \sec\epsilon_0$$

and

$$\tan B' = \tan B \sec\epsilon_0.$$

These latitudes are the latitudes of the points, where lines passing through the surface points perpendicular to the equator cut the sphere circumscribed with radius k round the centre of the ellipsoid, or also of the points where lines perpendicular to the polar axis meet the sphere circumscribed with radius $k\cos\epsilon_0$, so that we have

$$\left.\begin{array}{lr} \rho\sin\beta = \sin\beta'\cos\epsilon_0 \text{ and } & \cos\nu\sin B = \sin B'\cos\epsilon_0 \\ \rho\cos\beta = \cos\beta' & \cos\nu\cos B = \cos B' \\ & \sin\nu = \sin B'\sin\epsilon_0 \end{array}\right\} \quad . \quad . \quad . \quad (3)$$

and the equations (2) become

$$\left.\begin{array}{lcl} \rho\sin\sigma\sin\pi & = & \cos\beta'\sin(L-\lambda) \\ \rho\sin\sigma\cos\pi\,.\,\sec\epsilon & = & -\cos\beta'\cos(L-\lambda)\sin B' + \sin\beta'\cos B' \\ \rho\cos\sigma\,.\,\cos\nu & = & \cos\beta'\cos(L-\lambda)\cos B' + \sin\beta'\sin B'\,.\,\cos^2\epsilon_0 \end{array}\right\} \quad (4)$$

In order that Δ' should be a tangent to the ellipsoid, the coordinates of the surface point, if for a few moments we introduce rectangular coordinates

$$\begin{array}{ll} \xi = k\rho\cos\beta\cos\lambda & X = \Delta\cos B\cos L \\ \eta = k\rho\cos\beta\sin\lambda & Y = \Delta\cos B\sin L \\ \zeta = k\rho\sin\beta & Z = \Delta\sin B, \end{array}$$

must satisfy the equation

$$1 = \frac{\xi X + \eta Y}{kk} + \frac{\zeta Z}{kk\cos^2\epsilon_0},$$

or, returning to polar coordinates,

$$\frac{k}{\Delta} = \rho\cos\beta\cos(L-\lambda)\cos B + \rho\sin\beta\sin B\,.\,\sec^2\epsilon_0,$$

or finally

$$\frac{k}{\Delta}\cos\nu = \cos\beta'\cos(L-\lambda)\cos B' + \sin\beta'\sin B' \quad . \quad . \quad . \quad (5)$$

From this equation and the equations (4), the special values of $\rho\sin\sigma'$ and $\rho\cos\sigma'$ for points of the rim are to be deduced, which, substituted in the last equation (2), furnish the corresponding values of s' and Δ'.

The sum of the squares of the first and second equation (4) and of (5) gives

$$\rho^2 \sin^2 \sigma \,(\sin^2 \pi + \cos^2 \pi \sec^2 \epsilon) + \frac{k^2}{\Delta^2} \cos^2 \nu = 1$$

or

$$\rho^2 \sin^2 \sigma \,(1 + \tan^2 \epsilon \cos^2 \pi) = 1 - \frac{k^2}{\Delta^2} \cos^2 \nu.$$

Therefore, putting

$$\left.\begin{aligned} \frac{k}{\Delta} \cos \nu &= \sin h \\ \tan \epsilon \cos \pi &= \tan \epsilon' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we have

$$\rho \sin \sigma' = \cos h \cos \epsilon'.$$

The third equation (4) and (5) gives

$$\begin{aligned} \rho \cos \sigma' &= \frac{k}{\Delta} - \sin \beta' \sin B' \,.\, \sin^2 \epsilon_0 \sec \nu \\ &= \frac{k}{\Delta} - \tan^2 \epsilon_0 \,.\, \rho \sin \beta \sin B. \end{aligned}$$

But

$$\sin \beta = \sin \sigma \cos \pi \cos B + \cos \sigma \sin B.$$

Hence

$$\rho \cos \sigma' \,(1 + \tan^2 \epsilon_0 \sin^2 B) = \frac{k}{\Delta} - \tan^2 \epsilon_0 \,.\, \rho \sin \sigma' \cos \pi \cos B \sin B,$$

or as

$$\begin{aligned} \rho \sin \sigma' &= \cos h \cos \epsilon', \\ \rho \cos \sigma' &= \sin h \cos \nu - \cos h \sin \nu \sin \epsilon' \end{aligned}$$

Substituting these values of $\rho' \sin \sigma'$ and $\rho \cos \sigma'$ in the last equations (2) we derive

$$\left.\begin{aligned} &\Delta' \sec h \sin s' = k \cos \epsilon' \\ &\Delta' \sec h \cos s' = \Delta \cos h + k \sin \nu \sin \epsilon' \\ \text{or}\quad &\frac{\Delta' \cos \nu}{\Delta \cos h} \sin s' = \sin h \cos \epsilon' \\ &\frac{\Delta' \cos \nu}{\Delta \cos h} \cos s' = \cos h \cos \nu + \sin h \sin \nu \sin \epsilon' \\ &\tan s' = \frac{\tan h \sec \nu \cos \epsilon'}{1 + \tan h \tan \nu \sin \epsilon'} \\ \text{or, if preferred, indirectly}\quad & \\ &\tan s' = \frac{k \cos \epsilon'}{\Delta \cos h} (1 - \tan s' \sin \nu \tan \epsilon') \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

For

$\pi = \pm 90^\circ$ $\quad$ ϵ' is 0 hence $\tan s' = \tan h \sec \nu$ or $= \dfrac{k}{\Delta} \sec h$

$\pi = \dfrac{0^\circ}{180}$ $\quad$ $\epsilon' = \pm \epsilon$ hence $\tan s' = \dfrac{\tan h \sec \nu \cos \epsilon}{1 \pm \tan h \tan \nu \sin \epsilon}$

$$= \frac{k \cos \epsilon}{\Delta \cos h} (1 \mp \tan s' \sin \nu \tan \epsilon)$$

As ν has the same sign as B, the radii s' of the disc on that side of the equator, where the point of view is, are accordingly rather shorter than those on the opposite side.

If the values of λ and β of the points of the rim are required, they may be found by means of the equations

$$\left.\begin{aligned} \sec h \cos \beta' \sin (\mathrm{L}-\lambda) &= \cos \epsilon' \sin \pi \\ \sec h \cos \beta' \cos (\mathrm{L}-\lambda) &= -\frac{\cos \epsilon'}{\cos \epsilon} \cos \pi \sin \mathrm{B}' + \tan h \cos \mathrm{B}' \\ \sec h \sin \beta' &= \frac{\cos \epsilon'}{\cos \epsilon} \cos \pi \cos \mathrm{B}' + \tan h \sin \mathrm{B}' \end{aligned}\right\} . \quad (8)$$

The formulæ thus derived for the apparent disc of an ellipsoid may now be serviceable in the investigation of the shadow problem, the strict solution of which, when pursued in this way, will be found to offer no great obstacles, even if the light-supplying body is itself an ellipsoid of any eccentricity.

A few preliminary considerations may be useful for a clearer understanding of the question at issue. Let K be the radius of the (spherical) Sun, R the distance between Sun and planet, and let the point of view be in the prolongation of R and of the axis of the shadow. While the distance of the point from the centre of the planet is Δ, its distance from the Sun is $\mathrm{R}+\Delta$, so that the Sun's apparent disc is a circle of semi-diameter H, where $\sin \mathrm{H} = \dfrac{\mathrm{K}}{\mathrm{R}+\Delta}$, and upon this circle the dark disc of the planet is seen projected. Let now the point of view approach the planet or let Δ be lessened till the edges of the planet in position angles $\pm 90°$ touch the rim of the Sun. The Sun appears there in the shape of two cusps, the end points of which just touch, and the inner edges of which, being formed by the outline of the ellipsoid, are not symmetrical, except when B is 0, the cusp on the side of B being the broader one. By farther approaching the planet, the two discs cross, having tangents in common at the four crossing points. With Δ lessening the cusps diminish more and more, till the last rays of the Sun disappear, first on the side opposite to B, and finally at the end point of the disc's minor axis on the side of B. Now these groups of tangents, common to the two bodies at every point of the shadow axis within the indicated limits, form a continuous surface, the surface of the umbra, and it is required to determine the sections of this surface or the curves described by this surface on planes perpendicular to the shadow axis at any given points.

The angle s' between the tangent to the ellipsoid in position angle π and the shadow axis at a point at distance Δ, is found by

$$\tan s' = \frac{k \cos \epsilon'}{\Delta \cos h + k \sin \nu \sin \epsilon'},$$

the angle H between the tangent to the spherical Sun and the shadow axis by

$$\sin \mathrm{H} = \frac{\mathrm{K}}{\mathrm{R}+\Delta}.$$

In order that the tangent should be common to both bodies, the distance Δ must be so determined that $s' = \mathrm{H}$, or that

$$\sin s' = \frac{k \cos \epsilon' \cos s'}{\Delta \cos h + k \sin \nu \sin \epsilon'} = \frac{\mathrm{K}}{\mathrm{R}+\Delta},$$

and, therefore, that

$$\frac{k}{\Delta} = \frac{\mathrm{K} \cos h - k \cos \epsilon' \cos s'}{\mathrm{R} \cos \epsilon' \cos s' - \mathrm{K} \sin \nu \sin \epsilon'}.$$

Hence we get the value of h by

$$\tan h = \frac{\mathrm{K} \sec \epsilon' - k \sec h \cos s'}{\mathrm{R} \sec \nu \cos s' - \mathrm{K} \tan \nu \tan \epsilon'},$$

which, as it contains sec h and cos s', must be found indirectly if no conditions are introduced to make their differences from 1 insensible.

If r is the distance of a point of the tangent from the centre of the planet, and σ the angle between r and the shadow axis, the linear distance of the point from the axis will be

$$r \sin \sigma = (\Delta - r \cos \sigma) \tan s',$$

or as

$$\Delta \tan s' = \frac{k \cos \epsilon'}{\cos h + \sin h \tan \nu \sin \epsilon'},$$

$$r \sin \sigma = \frac{k - r \cos \sigma \,.\, \sin h \sec \nu}{\cos h \sec \epsilon' + \sin h \tan \nu \tan \epsilon'}.$$

The formulæ for finding the radii $r \sin \sigma$ of the curve formed by the umbra on a plane passing perpendicularly through the point $r \cos \sigma$ of the axis, are accordingly

$$\left.\begin{aligned} \tan \epsilon' &= \tan \epsilon \cos \pi \\ \tan h &= \frac{\mathrm{K} \sec \epsilon' - k \sec h \cos s'}{\mathrm{R} \sec \nu \cos s' - \mathrm{K} \tan \nu \tan \epsilon'} \\ \tan s' &= \frac{\tan h \sec \nu}{\sec \epsilon' + \tan h \tan \nu \tan \epsilon'} \\ r \sin \sigma &= \frac{k \sec h - r \cos \sigma \,.\, \tan h \sec \nu}{\sec \epsilon' + \tan h \tan \nu \tan \epsilon'} \end{aligned}\right\} \quad \ldots\ldots \quad (9)$$

Though the assumption that the illuminating body itself is an ellipsoid has at present no practical application, I will not fail to give also the formulæ adapted to that case.

Let K be the equatorial radius and $\mathrm{E} = \sin \eta_0$ the eccentricity of the ellipsoidal Sun, *B* the latitude above the Sun's equator and $\mathrm{R} + \Delta$ the distance of the point of view in the shadow axis, and

let the position angle of the polar axis of the planet, reckoned from the direction of the polar axis of the Sun, be P, so that the position angle of a tangent to the Sun corresponding to π is $\pi + P$.

Making then

$$\tan \gamma = \tan \eta_0 \,.\, \sin B$$
$$\tan \eta = \tan \eta_0 \,.\, \cos B \sec \gamma$$
$$\tan \eta' = \tan \eta \cos (\pi - P)$$

and

$$\sin H = \frac{K \cos \gamma}{R + \Delta},$$

the angle H′ between the tangent in position angle $\pi + P$ and the shadow axis will be

$$\tan H' = \frac{K \cos \eta'}{(R + \Delta) \cos H + K \sin \gamma \sin \eta'},$$

and this tangent will be common to both bodies, planet and Sun, if Δ is so determined that

$$\frac{k \cos \epsilon'}{\Delta \cos h + k \sin \nu \sin \epsilon'} = \frac{K \cos \eta'}{(R + \Delta) \cos H + K \sin \gamma \sin \eta'}.$$

Accordingly the value of

$$\frac{k \cos \nu}{\Delta \cos h} = \tan h$$

is found by

$$\tan h = \frac{K \cos \eta' \sec \epsilon' - k \cos H \sec h}{R \cos H + K \cos \eta' (\sin \gamma \tan \eta' - \sin \nu \tan \epsilon')} \,.\, \cos \nu$$

(the cos H involved being derived from

$$\left. \sin H = \frac{K \cos \gamma}{R + \frac{\sin h \sec \nu}{k}} \right).$$

and finally the radius $r \sin \sigma$ in position angle π of the curve formed by the surface of the umbra on a plane passing perpendicularly through the point $r \cos \sigma$ of the shadow axis

$$r \sin \sigma = \frac{k \sec h - r \cos \sigma \,.\, \tan h \sec \nu}{\sec \epsilon' + \tan h \sec \nu \,.\, \sin \nu \tan \epsilon'}.$$

The application of these formulæ to fictitious cases leads to a variety of interesting curves, the consideration of which I must here forego.

In the cases of *Jupiter* and of *Saturn* the computation of the sections of their shadows becomes simplified. The "Ephemerides for Physical Observations of *Jupiter*" for 1891 and succeeding years contain the longitudes $\Lambda + 180°$ and latitudes B of the Sun

(referred to the plane of *Jupiter's* equator and corrected for aberration), which, when used with the "Data for Computing the Positions of the Satellites of *Jupiter*," serve in computing the jovicentric coordinates

$$r \sin\sigma \sin\pi, \qquad r \sin\sigma \cos\pi, \qquad r \cos\sigma$$

of the satellites referred to the axis of the shadow cone as the z axis, as mentioned in *Monthly Notices*, vol. li. p. 508. The distance $r \sin\sigma$ of the satellite from the axis of the shadow cone must then be compared with the corresponding radius of the section of the shadow which passes through the point $r \cos\sigma$ of the z axis. The formulæ become ($-B$ being substituted for B)

$$\tan\nu = \tan\epsilon_0 \sin B$$

$$\tan\epsilon = \tan\epsilon_0 \cos B \sec\nu$$

$$\tan\epsilon' = \tan\epsilon \cos\pi$$

$$h = \frac{\mathrm{K}\sec\epsilon' - k}{\mathrm{R} - \mathrm{K}\sin\nu\tan\epsilon'}$$

(h being written for $\tan h \sec\nu$)

$$r \sin\sigma = \frac{k - h \,.\, r\cos\sigma}{\sec\epsilon' - h \,.\, \sin\nu\tan\epsilon'}.$$

In the case of the shadow of the Earth the formulæ become still simpler so far as the geometrical shadow is concerned.

Let A D be the right ascension and declination of the axis of the shadow, or of the prolongation of R, α δ the geocentric right ascension and declination of the Moon, π its parallax, or $r = \dfrac{1}{\sin\pi}$ its linear distance, and κ its semi-diameter expressed in semi-diameters of the Earth's equator. Further, let a d be the right ascension and declination of the Moon's centre for a spot on its surface in selenographical longitude λ and latitude β, N the node and J the inclination of the Moon's equator referred to the plane parallel to the Earth's equator, and α_0 the Moon's mean right ascension, analogous to l_0 its mean longitude ($\alpha_0 - \mathrm{N} = l_0 - \Omega + \Delta + 180$ in Encke's notation, adopted in the *Nautical Almanac*, N and J being the Ω' and i, v. *Monthly Notices*, vol. xli. p. 420 ff.) The formulæ for finding the coordinates of the spot λ β referred to the shadow axis are

$$\sec\beta\cos d\sin(a - \mathrm{N}) = \sin(\alpha_0 - \mathrm{N} + \lambda)\cos\mathrm{J} + \tan\beta\sin\mathrm{J}$$

$$\sec\beta\cos d\cos(a - \mathrm{N}) = \cos(\alpha_0 - \mathrm{N} + \lambda)$$

$$\sec\beta\sin d \qquad = \sin(\alpha_0 - \mathrm{N} + \lambda)\sin\mathrm{J} - \tan\beta\cos\mathrm{J}$$

$$x = r\cos\delta\sin(\alpha - \mathrm{A}) \qquad \xi = \kappa\cos d\sin(a - \mathrm{A})$$

$$y = r\sin(\delta - \mathrm{D}) + x\tan\tfrac{1}{2}(\alpha - \mathrm{A})\sin\mathrm{D} \qquad \eta = \kappa\sin(d - \mathrm{D}) + \xi\tan\tfrac{1}{2}(a - \mathrm{A})\sin\mathrm{D}$$

$$z = r\cos(\delta - \mathrm{D}) - x\tan\tfrac{1}{2}(\alpha - \mathrm{A})\cos\mathrm{D} \qquad \zeta = \kappa\cos(d - \mathrm{D}) - \xi\tan\tfrac{1}{2}(\alpha - \mathrm{A})\cos\mathrm{D}$$

Hence the position angle p of the spot, its angular distance σ from the shadow axis and its linear distance r' from the Earth's centre are

$$r' \sin \sigma \sin p = x - \xi$$
$$r' \sin \sigma \cos p = y - \eta$$
$$r' \cos \sigma \qquad = z - \zeta.$$

If $\cos \epsilon_0$ is the proportion of the Earth's polar axis to the diameter of the equator

$$\sin \epsilon = \sin \epsilon_0 \cos D$$
$$\tan \epsilon' = \tan \epsilon \cos p$$
$$h = \frac{K - k}{R},$$

the radius of the section of the geometrical umbra is

$$r' \sin \sigma = \cos \epsilon' - h \,.\, r' \cos \sigma.$$

But the observed enlargement of the shadow proves that while a lunar spot near the border of the shadow receives still, or receives already, direct sunlight from a cusp outside the Earth's atmosphere, this illumination is so feeble that, as seen from the Earth, it merges into the dim illumination produced by the broken sunlight passing through the atmosphere and surrounding part of the Earth's disc. The uncertainty in fixing upon what ought to be considered the true border-line of the full shadow leaves it doubtful how much of the light near the margin of the Sun is to be treated as practically ineffective, or how much the value of K must be diminished in order to furnish the border-line. If I am not prevented from preparing ephemerides for physical observations of the Moon for 1898, I intend to give, for a number of lunar spots, predictions of the approximate times of their crossing the border-line of the shadow during the three lunar eclipses of that year, in the hope that many observers may join in making contributions for the closer examination of the question.

Col. Cooper's Observatory, Markree, Collooney, Ireland.

The Orbit of δ Cygni. By S. W. Burnham.

The companion to δ *Cygni* was discovered by Sir William Herschel on 1783 September 20. The angle was measured, but all we know of the distance of the components at that time is found in his description of the appearance of the star in his reflector. Herschel says, "With 278 about $\frac{1}{2}$-diameter of L [large star]; with 460 $\frac{3}{4}$-diameter of L; with 932 full $1\frac{1}{2}$-diameter of L in hazy weather, which has taken off the rays of